Consider the nonlinear Klein-Gordon equation

$$\begin{aligned}
&\varepsilon^2 \partial_{tt} u(x,t) - \partial_{xx} u(x,t) + \frac{1}{\varepsilon^2} u(x,t) + f(u(x,t)) = 0, \quad 0 < x < 1,\ 0 < t < T, \\
&u(x,0) = g_0(x), \qquad \partial_t u(x,0) = \frac{1}{\varepsilon^2} g_1(x), \qquad 0 \le x \le 1, \\
&u(0,t) = u(1,t) = 0, \qquad 0 \le t \le T,
\end{aligned}$$

where $0 < \varepsilon \le 1$ is a given dimensionless constant, $f(u)$ is a function of u and $g_0(x)$ and $g_1(x)$ are given functions, which are all independent of ε.

1. Define the Hamiltonian (or energy) as

$$E(t) := \int_0^1 \left[\varepsilon^2 |\partial_t u|^2 + |\partial_x u|^2 + \frac{1}{\varepsilon^2} u^2 + F(u) \right] dx, \qquad t \ge 0,$$

   where

$$F(u) = 2 \int_0^u f(s)\, ds.$$

   Show that the Hamiltonian is conserved, i.e.

$$E(t) \equiv E(0), \qquad t \ge 0.$$

2. Construct an explicit second-order (in space and time) finite difference (EXFD) method for the problem and find its linear stability.

3. Construct a second-order (in space and time) finite difference method for the problem such that the the Hamiltonian (or energy) is conserved in the discretized level and prove it.